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Nonradial large solutions of sublinear elliptic problems

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Abstract

Let p be a nonnegative locally bounded function on $\mathbb{R}^N$, $N \geqslant 3$, and $0 < \gamma < 1$. Assuming that the oscillation $\sup_{|x|=r} p(x) - \inf_{|x|=r} p(x)$ tends to zero as $r \to \infty$ at a specified rate, it is shown that the equation $\Delta u = p(x)u^\gamma$ admits a positive solution in $\mathbb{R}^N$ satisfying $\lim_{|x|\to\infty} u(x) = \infty$ if and only if

$$\int\limits_{\mathbb{R}^N} \frac{p(x)}{|x|^{N-2}}\, dx = \infty.$$

Keywords: Large solution; Semilinear Dirichlet problem; Greenian domain; Green function

1. Introduction

We are interested in positive solutions to the boundary blow-up problem

$$\Delta u = p(x)u^\gamma \quad \text{in } \Omega, \tag{1.1}$$

$$u = \infty \quad \text{on } \partial\Omega, \tag{1.2}$$

where $0 < \gamma < 1$, Ω is a domain of $\mathbb{R}^N$, and p is nonnegative locally bounded function on Ω. Such a solution is called a *large* solution to the sublinear elliptic equation (1.1) in Ω. In the case $\Omega = \mathbb{R}^N$, a large solution is called an *entire large* solution. For bounded domains Ω,

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the boundary condition (1.2) means that $\lim_{x\in\Omega,\, x\to z} u(x) = \infty$ for every $z \in \partial\Omega$. If Ω is unbounded, we require moreover that $\lim_{x\in\Omega,\, |x|\to\infty} u(x) = \infty$.

In the situation $\gamma > 1$, boundary blow-up problems of the kind (1.1)–(1.2) have received much attention during the last years (see [2,5–8,11,13,16] and their references). For $0 < \gamma \leqslant 1$ and in the case where $p(x) = p(|x|)$ is radial on $\mathbb{R}^N$, $N \geqslant 3$, Lair and Wood [14] proved that (1.1) possesses an entire large solution (which is radial) if and only if

$$\int_0^\infty r p(r)\, dr = \infty.$$

Later, Lair [12] considered problems of the type (1.1)–(1.2) for continuous functions p which are not necessarily radial on $\mathbb{R}^N$. Assuming that

$$\int_0^\infty r p_{\mathrm{osc}}(r) \exp\bigl(\psi(r)\bigr)\, dr < \infty, \tag{1.3}$$

where

$$\psi(r) = \int_0^r t \inf_{|x|=t} p(x)\, dt \quad \text{and} \quad p_{\mathrm{osc}}(r) = \sup_{|x|=r} p(x) - \inf_{|x|=r} p(x),$$

he proved that (1.1) has an entire large solution if and only if

$$\int_0^\infty r \inf_{|x|=r} p(x)\, dr = \infty. \tag{1.4}$$

The question raised in [14] and [12], if even for nonradial functions p condition (1.4) alone is sufficient for the existence of an entire large solution to (1.1) remained open.

In this paper, we shall show that, in general, (1.4) is neither necessary nor sufficient for (1.1) to have an entire large solution. But if the oscillation function $p_{\mathrm{osc}}(r)$ is small enough in the sense that

$$\int_0^\infty r p_{\mathrm{osc}}(r) \bigl(1 + \psi(r)\bigr)^{\gamma/(1-\gamma)}\, dr < \infty \tag{1.5}$$

(which is clearly weaker than (1.3)), then Eq. (1.1) admits an entire large solution if and only if (1.4) holds (Theorem 5.1). Moreover, if (1.5) holds, then

$$(1.4) \quad \Leftrightarrow \quad \int_0^\infty r \sup_{|x|=r} p(x)\, dr = \infty \quad \Leftrightarrow \quad \int_{\mathbb{R}^N} \frac{p(x)}{|x|^{N-2}}\, dx = \infty.$$

In the case, where Ω is the unit open ball, problem (1.1)–(1.2) is studied in [7] for radial functions p such that $c_1 \leqslant (1-r)^\alpha p(r) \leqslant c_2$ for all $0 \leqslant r < 1$, where c_1, c_2 and α are positive constants. We consider problem (1.1)–(1.2) for Greenian domains Ω of $\mathbb{R}^N$, $N \geqslant 1$, and establish some nonexistence results of large solutions to (1.1) (see Section 3). It will be shown that problem (1.1)–(1.2) has no positive solutions if (1.1) admits a positive bounded solution in Ω. In particular, for $N \geqslant 3$, Eq. (1.1) does not possess an entire large solution provided

$$\int_{\mathbb{R}^N} \frac{p(x)}{|x|^{N-2}}\,dx < \infty.$$

We note that the problem of the existence of bounded positive solutions to (1.1) is discussed in [10], where necessary and sufficient conditions are given.

2. The operator U_D^p

For every open set $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, let $\mathcal{B}(\Omega)$ be the set of all Borel measurable functions from Ω to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and let $\mathcal{C}(\Omega)$ be the set of all continuous real-valued functions on Ω. If $\mathcal{F}$ is a set of numerical functions then $\mathcal{F}^+$ (respectively $\mathcal{F}_b$) will denote the class of all functions in $\mathcal{F}$ which are nonnegative (respectively bounded).

An open relatively compact subset D of $\mathbb{R}^N$ (we shall write $D \Subset \mathbb{R}^N$) is called *regular* (for Δ), if each function $f \in \mathcal{C}(\partial D)$ admits a continuous extension $H_D f$ on $\overline{D}$ such that $H_D f$ is harmonic on D. In other words, the function $h = H_D f$ is the unique solution to the classical Dirichlet problem

$$\begin{cases} \Delta h = 0 & \text{in } D, \\ h = f & \text{on } \partial D. \end{cases}$$

For every $x \in D$, the *harmonic measure* relative to x and D, which will be denoted by μ_x^D, is defined to be the positive Radon measure on ∂D given by the mapping $f \mapsto H_D f(x)$. For every $f \in \mathcal{B}(\mathbb{R}^N)$ we define

$$H_D f(x) := \begin{cases} \int f\,d\mu_x^D & \text{if } x \in D, \\ f(x) & \text{if } x \in \mathbb{R}^N \setminus D, \end{cases}$$

provided the integral makes sense. H_D is called the *harmonic kernel* on D.

A lower semi-continuous function $s > -\infty$ is said to be *superharmonic* on the open set $\Omega \subset \mathbb{R}^N$ if $H_D s$ is harmonic on D and $H_D s \leqslant s$ for all regular open sets $D \Subset \Omega$. If, moreover, the constant zero is the only nonnegative harmonic function majorized by s on Ω, we call s a *potential* on Ω. Every function v such that $-v$ is superharmonic on Ω will be called *subharmonic* on Ω. Clearly, h is harmonic on Ω if and only if it is both superharmonic and subharmonic on Ω.

An open subset Ω of $\mathbb{R}^N$ is called a *Greenian set* if Ω possesses a *Green function* which will be denoted by G_Ω. For every $y \in \Omega$, $G_\Omega(\cdot, y)$ is a potential on Ω and we have $-\Delta G_\Omega(\cdot, y) = \varepsilon_y$ where ε_y denotes the Dirac measure at the point y. In the case of $N \leqslant 2$, $\mathbb{R}^N$ is not Greenian but all bounded open subsets are Greenian. If $N \geqslant 3$ then every open subset of $\mathbb{R}^N$ is Greenian, in particular the whole space $\mathbb{R}^N$ is Greenian and it is well known that for every $x, y \in \mathbb{R}^N$

$$G_{\mathbb{R}^N}(x, y) = G(x, y) := \begin{cases} c_N |x - y|^{2-N} & \text{if } x \neq y, \\ \infty & \text{if } x = y, \end{cases} \tag{2.1}$$

where $c_N > 0$ is a constant depending only on N. For every Greenian set Ω we define $G_\Omega u$ on Ω by

$$G_\Omega u(x) := \int_\Omega G_\Omega(x, y) u(y)\,dy \tag{2.2}$$

for every $u \in \mathcal{B}(\Omega)$ for which the integral makes sense. Note that if $u \geqslant 0$ and Ω is connected, then $G_\Omega u$ is either a potential or identically ∞ on Ω. If $u \in \mathcal{B}_b(D)$ where $D \Subset \mathbb{R}^N$ is an open set, then $G_D u \in \mathcal{C}_b(D)$. If, moreover, D is regular, then $\lim_{x \to z} G_D u(x) = 0$ for all $z \in \partial D$.

In the following, let p be a nonnegative function in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and let γ be a real number such that $0 < \gamma \leqslant 1$. By a solution to the equation

$$\Delta u = p(x) u^\gamma \tag{2.3}$$

in an open set $\Omega \subset \mathbb{R}^N$, we shall mean every real-valued continuous function $u \geqslant 0$ on Ω satisfying (2.3) in the distributional sense, i.e., pu^γ is locally (Lebesgue) integrable on Ω and the equality

$$\int_\Omega u(x) \Delta\varphi(x)\, dx = \int_\Omega p(x) u^\gamma(x) \varphi(x)\, dx$$

holds for every nonnegative function φ belonging to the space $\mathcal{C}_c^\infty(\Omega)$ of infinitely differentiable functions on Ω with compact support. Supersolutions and subsolutions of (2.3) are to be understood in the same way replacing "$=$" by "$\leqslant$" and "$\geqslant$," respectively. Analogously, we also define solutions, subsolutions, and supersolutions to nonlinear equations of the kind

$$\Delta u = p(x) g(u)$$

where g is a Borel measurable function on $\mathbb{R}$. We state the following three useful results which have been proved in [10].

Lemma 2.1. *Let $g \in \mathcal{B}(\mathbb{R})$ be a nondecreasing function, $D \Subset \mathbb{R}^N$ be an open subset and let $u, v \in \mathcal{C}(D)$. Assume that*

$$\Delta u \leqslant p(x) g(u), \qquad \Delta v \geqslant p(x) g(v) \quad \text{in } D$$

and $\liminf_{x \to z}(u - v)(x) \geqslant 0$ *for all $z \in \partial D$. Then $u \geqslant v$ in D.*

Lemma 2.2. *The following holds*

(a) *Let D be a bounded open subset of $\mathbb{R}^N$ and $u \in \mathcal{B}^+(D)$ be bounded. Then u is a solution of* (2.3) *in D if and only if $u + G_D(pu^\gamma)$ is harmonic on D.*
(b) *Let Ω be an open set of $\mathbb{R}^N$ and $u \in \mathcal{B}^+(\Omega)$ be locally bounded. Then u is a solution of* (2.3) *in Ω if and only if $u + G_D(pu^\gamma) = H_D u$ for every regular open set $D \Subset \Omega$.*

Lemma 2.3. *For every regular open set $D \Subset \mathbb{R}^N$ and every $f \in \mathcal{C}^+(\partial D)$, there exists one and only one $u \in \mathcal{C}^+(\overline{D})$, which will be denoted by $U_D^p f$, such that*

$$\begin{cases} \Delta u = p(x) u^\gamma & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases} \tag{2.4}$$

It will be convenient to identify real functions on a set $A \subset \mathbb{R}^N$ with functions on $\mathbb{R}^N$ which have value zero outside A. Let f be a nonnegative continuous function on $\mathbb{R}^N$. If D is a regular open subset of $\mathbb{R}^N$, we extend $U_D^p f$ on $\mathbb{R}^N \setminus D$ by $U_D^p f(x) = f(x)$ for every $x \in \mathbb{R}^N \setminus D$. Obviously, U_D^p coincides with the harmonic kernel H_D if p is identically zero on D. In general, however, the operator U_D^p is not linear. In the following proposition, we collect some useful properties of U_D^p.

Proposition 2.4. *The following hold*:

(a) *Let $D \Subset \mathbb{R}^N$ be a regular open set and let $f, g \in \mathcal{C}^+(\partial D)$. Then*:
  (a1) *$u := (U_D^p f)|_D$ is the unique solution of the integral equation*

$$u + \int_D G_D(\cdot, y) p(y) u^\gamma(y)\, dy = H_D f. \tag{2.5}$$

  (a2) *U_D^p is nondecreasing on $\mathcal{C}^+(\partial D)$: If $f \geqslant g$ on ∂D, then $U_D^p f \geqslant U_D^p g$.*
  (a3) *U_D^p is convex on $\mathcal{C}^+(\partial D)$: For every $0 \leqslant \alpha \leqslant 1$,*

$$U_D^p\big(\alpha f + (1-\alpha) g\big) \leqslant \alpha U_D^p f + (1-\alpha) U_D^p g.$$

  In particular, $U_D^p(\lambda f) \geqslant \lambda U_D^p f$ for all $\lambda \geqslant 1$.

(b) *Let Ω be an open subset of $\mathbb{R}^N$ and let $u \in \mathcal{C}^+(\Omega)$. Then*:
  (b1) *The function u is a solution* (*respectively supersolution, subsolution*) *to* (2.3) *in Ω if and only if for every regular open set $D \Subset \Omega$*

$$U_D^p u = u \quad \big(\textit{respectively} \quad U_D^p u \leqslant u,\ U_D^p u \geqslant u\big).$$

  (b2) *If u is a supersolution* (*respectively subsolution*) *to* (2.3) *in Ω, then*

$$U_D^p u \leqslant U_{D'}^p u \quad \big(\textit{respectively} \quad U_D^p u \geqslant U_{D'}^p u\big)$$

  for all regular open sets D, D' such that $D' \Subset D \Subset \Omega$.

(c) *Let $q \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ such that $0 \leqslant q \leqslant p$. Then $U_D^p f \leqslant U_D^q f$ for every regular open set $D \Subset \mathbb{R}^N$ and every $f \in \mathcal{C}^+(\partial D)$.*

Proof. Statement (a1) follows from (b) in Lemma 2.2. The comparison principle, given by Lemma 2.1, implies that U_D^p is nondecreasing on $\mathcal{C}^+(\partial D)$. Furthermore, the same lemma yields statement (a3). Indeed,

$$\begin{aligned}\Delta\big(\alpha U_D^p f + (1-\alpha) U_D^p g\big) &= p(x)\big(\alpha\big(U_D^p f\big)^\gamma + (1-\alpha)\big(U_D^p g\big)^\gamma\big)\\ &\leqslant p(x)\big(\alpha U_D^p f + (1-\alpha) U_D^p g\big)^\gamma.\end{aligned}$$

The last part in (a3) is obvious since $U_D^p 0 = 0$. Again the comparison principle yields (b1). To obtain (b2) it suffices to use (b1) and (a2). Finally to get (c) it is enough to see that $U_D^q f$ is a supersolution to Eq. (2.3) in D. □

The following Harnack type inequality has been shown in [3]. For the convenience of the reader we include a proof.

Proposition 2.5. *Let Ω be a domain in $\mathbb{R}^N$. For every compact set $K \subset \Omega$ there exists a constant $C > 0$ such that*

$$\sup_{x \in K} u(x) \leqslant C\Big(\inf_{x \in K} u(x) + 1\Big)$$

for every nonnegative solution u of (2.3) *in Ω.*

Proof. Let $K \subset \Omega$ be a compact set and let $a \in K$. By [4, Proposition 7.6], there exists $r > 0$ such that $B := B(a, r) \Subset \Omega$ and the inequality

$$\int_B G_B(x,z)p(z)h(z)\,dz \leqslant \frac{1}{2(1+\gamma)}h(x) \tag{2.6}$$

holds for every $x \in B$ and every nonnegative harmonic function h on B. Let u be a nonnegative solution to Eq. (2.3) in Ω. Then, by (a1) in Proposition 2.4

$$u + \int_B G_B(\cdot,z)p(z)u^\gamma(z)\,dz = H_B u \quad \text{in } B.$$

Let $C_1 := (1-\gamma)\sup_{x\in B}\int_B G_B(x,z)p(z)\,dz < \infty$. Using (2.6) and the inequality $a^\gamma \leqslant \gamma a + 1 - \gamma$ for $a \geqslant 0$, we obtain that, for every $y \in B$,

$$\begin{aligned} H_B u(y) &\leqslant u(y) + \int_B G_B(y,z)p(z)\big(H_B u(z)\big)^\gamma\,dz \\ &\leqslant u(y) + \gamma\int_B G_B(y,z)p(z)H_B u(z)\,dz + (1-\gamma)\int_B G_B(y,z)p(z)\,dz \\ &\leqslant u(y) + \frac{1}{2}H_B u(y) + C_1. \end{aligned}$$

This yields that $H_B u(y) \leqslant 2u(y) + 2C_1$ for all $y \in B$. On the other hand, by the classical Harnack principle (see for instance [9]), there exists a constant $C_0 > 0$ such that the inequality

$$\sup_{x\in B(a,r/2)} h(x) \leqslant C_0 \inf_{x\in B(a,r/2)} h(x)$$

holds for all nonnegative harmonic functions h in B. Therefore, for all $x, y \in B(a, r/2)$,

$$u(x) \leqslant H_B u(x) \leqslant C_0 H_B u(y) \leqslant 2C_0 u(y) + 2C_0 C_1 \leqslant C_2\big(u(y)+1\big)$$

where $C_2 := 2C_0 \max(1, C_1)$. Thus, for every $a \in K$, there exist $\rho_a, C_a > 0$ such that $B(a,\rho_a) \Subset \Omega$ and

$$\sup_{x\in B(a,\rho_a)} u(x) \leqslant C_a\Big(\inf_{x\in B(a,\rho_a)} u(x) + 1\Big)$$

for every nonnegative solution u of (2.3) in Ω. Now, since K is compact, we may choose a finite subset A of K such that $K \subset \bigcup_{a\in A} B(a,\rho_a)$. To finish the proof it suffices to take $C = \max_{a\in A} C_a$. $\square$

We shall call a (*regular*) *exhaustion* of the open set Ω every sequence of (regular) open sets $(\Omega_n)_{n\geqslant 1}$ satisfying

$$\Omega_n \Subset \Omega_{n+1} \quad \text{for all } n \geqslant 1 \quad \text{and} \quad \bigcup_{n\geqslant 1}\Omega_n = \Omega. \tag{2.7}$$

Obviously, every sequence of concentric balls with radii tending to infinity is a regular exhaustion of $\mathbb{R}^N$. More generally, a regular exhaustion of a regular Greenian domain $\Omega \subset \mathbb{R}^N$ can be constructed as follows: Take a fixed point $x_0 \in \Omega$ and define

$$\Omega_n := \big\{x \in \Omega:\ G_\Omega(x_0,x) > \alpha_n\big\},$$

where $\alpha_n > \alpha_{n+1} > 0$ for every $n \geqslant 1$ and $\lim_{n\to\infty}\alpha_n = 0$. We note that, by Corollary 6.6.13 in [1], every open (not necessarily regular) subset Ω of $\mathbb{R}^N$ admits a regular exhaustion $(\Omega_n)_{n\geqslant 1}$. Moreover, Ω_n may be chosen to be connected if Ω is connected.

Proposition 2.6. *Let Ω be an open subset of $\mathbb{R}^N$ and let (u_n) be a sequence of nonnegative solutions to Eq.* (2.3) *in Ω which is locally uniformly bounded on Ω. Then there exists a subsequence of (u_n) which is locally uniformly convergent on Ω.*

Proof. Let K be a compact set and let D be an open set such that $K \subset D \Subset \Omega$. For every $n \geqslant 1$, we define

$$h_n(x) := u_n(x) + \int_D G_D(x, y) u_n^\gamma(y) p(y)\, dy \qquad (x \in D).$$

Since (h_n) is a uniformly bounded sequence of harmonic functions in D, we may choose a subsequence (h_{n_k}) of (h_n) which is uniformly convergent on K (see [1, Theorem 1.5.11]). On the other hand, it is well known that G_D, given by (2.2), is a compact operator on $\mathcal{B}_b(D)$ endowed with the uniform norm. Therefore, (u_{n_k}) admits a uniformly convergent subsequence on K. We then conclude that for every compact subset K of Ω there exists a subsequence of (u_n) which converges uniformly on K. Hence, we consider a regular exhaustion (Ω_k) of Ω and we apply the diagonal procedure to construct a subsequence of (u_n) which is uniformly convergent on Ω_k for all $k \geqslant 1$. $\square$

Corollary 2.7. *Let (u_n) be a sequence of nonnegative solutions to Eq.* (2.3) *in a domain Ω of $\mathbb{R}^N$.*

(a) *If $\sup_{n \geqslant 1} u_n(x_0) < \infty$ for some point $x_0 \in \Omega$, then (u_n) has a locally uniformly convergent subsequence on Ω.*
(b) *If (u_n) is pointwise convergent to a function u, then u is a solution of* (2.3) *in Ω and the convergence is uniform on every compact subset of Ω.*

Proof. Statement (a) is a direct consequence of Propositions 2.5 and 2.6. Statement (b) is trivial. $\square$

3. Nonexistence results

We call an *entire solution* every solution to Eq. (2.3) in the whole space $\mathbb{R}^N$. An entire solution u is said to be *large* provided

$$\lim_{|x| \to \infty} u(x) = \infty.$$

More generally, for every bounded domain $\Omega \subset \mathbb{R}^N$, we say that a nonnegative solution u of (2.3) in Ω is large if $\lim_{x \in \Omega, x \to z} u(x) = \infty$ for every $z \in \partial\Omega$, but we require moreover that $\lim_{x \in \Omega, |x| \to \infty} u(x) = \infty$ if Ω is not bounded. In this section, we discuss some sufficient conditions under which Eq. (2.3) cannot admit a nonnegative large solution.

Theorem 3.1. *Let Ω be a Greenian domain of $\mathbb{R}^N$, $N \geqslant 2$, and let p be a nonnegative function in $L^\infty_{\mathrm{loc}}(\Omega)$. Then Eq.* (2.3) *has no large solutions in Ω if it has a nontrivial nonnegative bounded solution in Ω.*

Proof. We consider a regular exhaustion $(D_n)_{n\geqslant 1}$ of Ω and let $\lambda \geqslant 1$. Since positive constants are supersolutions to Eq. (2.3), the sequence $(U^p_{D_n}\lambda)_{n\geqslant 1}$ is nondecreasing and the limit function

$$u_\lambda := \lim_{n\to\infty} U^p_{D_n}\lambda$$

is a nontrivial solution of (2.3) such that $0 \leqslant u_\lambda \leqslant \lambda$ (see [10, Lemma 3]). Moreover, by (a3) in Proposition 2.4, we have

$$U^p_{D_n}\lambda \geqslant \lambda U^p_{D_n}1$$

for all $\lambda \geqslant 1$ and all $n \geqslant 1$. Therefore, letting n tend to ∞ we obtain that $u_\lambda \geqslant \lambda u_1$ for every $\lambda \geqslant 1$. Suppose now that (2.3) has a large solution u in Ω. Then, by Lemma 2.1, $u \geqslant u_\lambda$ and thereby $u \geqslant \lambda u_1$ in Ω for every $\lambda \geqslant 1$. This yields a contradiction because u_1 is not identically zero in Ω. □

A subset A of $\mathbb{R}^N$ is called *thick* (or 1-*thick*) if each nonnegative superharmonic function s on $\mathbb{R}^N$ has the following property:

$$s \geqslant 1 \quad \text{on } A \quad \Rightarrow \quad s \geqslant 1 \quad \text{on } \mathbb{R}^N. \tag{3.1}$$

More generally, a subset A of a Greenian domain Ω is called 1-*thick* (with respect to Ω) if, replacing $\mathbb{R}^N$ by Ω, (3.1) holds true for every nonnegative superharmonic function s on Ω. Note that for $\Omega = \mathbb{R}^N$, a set A is nonthick if and only if A is *thin at* ∞ in the sense of [1] or [9], which in turn means that A is *recurrent* in the sense of [15].

The first author proved in [10] that Eq. (2.3) has a nontrivial nonnegative bounded solution in a Greenian domain Ω of $\mathbb{R}^N$, $N \geqslant 1$, if and only if we may find a point $x_0 \in \Omega$ and a non-1-thick Borel set $A \subset \Omega$ such that

$$\int_{\Omega\setminus A} G_\Omega(x_0, y)p(y)\,dy < \infty.$$

For $\Omega = \mathbb{R}^N$ with $N \geqslant 3$, it is not hard to see that the above condition holds true (with $A = \emptyset$) whenever

$$\int_0^\infty r p^*(r)\,dr < \infty, \tag{3.2}$$

where p^* is defined for every $r \geqslant 0$ by $p^*(r) = \sup_{|x|=r} p(x)$. The converse statement is also shown in the same paper [10] provided the function p is radially symmetric on $\mathbb{R}^N$ (see [10, Theorem 1]). The following corollaries are immediate consequences of Theorem 3.1.

Corollary 3.2. *Let p be a nonnegative function in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, $N \geqslant 3$. Under each of the following conditions Eq.* (2.3) *has no entire large solution in $\mathbb{R}^N$.*

(A$_1$) *The set $\{p > 0\}$ is thin at ∞.*
(A$_2$) *There exists a point $x_0 \in \mathbb{R}^N$ such that*

$$\int_{\mathbb{R}^N} \frac{p(y)}{|x_0 - y|^{N-2}}\,dy < \infty. \tag{3.3}$$

(A_2') *There exists $\eta \geqslant 0$ such that*

$$\int_{\eta}^{\infty} r p^*(r)\,dr < \infty.$$

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a Greenian domain and let p be a nonnegative function in $L^\infty_{\mathrm{loc}}(\Omega)$. Under each of the following conditions Eq.* (2.3) *has no large solution in Ω.*

(B_1) *The set $\{p > 0\}$ is non-*1*-thick.*
(B_2) *There exists a point $x_0 \in \mathbb{R}^N$ such that*

$$\int_{\Omega} G_\Omega(x_0, y)p(y)\,dy < \infty. \tag{3.4}$$

4. Radial perturbation

In this section, we investigate Eq. (2.3) in the case where the function p is radially symmetric on $\mathbb{R}^N$. Our main goal here is to give necessary and sufficient conditions for the existence of an entire large solution to (2.3) in $\mathbb{R}^N$ for $N \geqslant 3$ (see Theorem 4.5). First, we need the following lemmas.

Lemma 4.1. *Let $D \Subset \mathbb{R}^N$ be a regular domain and let x_0 be a fixed point in D. Then the function z given by $z(\lambda) := U_D^p\lambda(x_0)$ for every nonnegative real λ has the following properties*:

(a) $0 \leqslant z(\lambda) - z(\mu) \leqslant \lambda - \mu$ *for every* $\lambda \geqslant \mu \geqslant 0$.
In particular, z is continuous and nondecreasing in $[0, \infty[$.
(b) $z(0) = 0$ *and* $\lim_{\lambda\to\infty} z(\lambda) = \infty$.

Proof. By (2.5),

$$z(\lambda) - z(\mu) + \int_D G_D(x_0, y)\big(\big(U_D^p\lambda(y)\big)^\gamma - \big(U_D^p\mu(y)\big)^\gamma\big)p(y)\,dy = \lambda - \mu,$$

where $U_D^p\mu \leqslant U_D^p\lambda$ if $0 \leqslant \mu \leqslant \lambda$. Therefore (a) holds.

Obviously, $z(0) = 0$. To show that $\lim_{\lambda\to\infty} z(\lambda) = \infty$ we fix $x \in D$ such that $U_D^p 1(x) > 0$. By Proposition 2.5 and (a3) in Proposition 2.4, for every $\lambda \geqslant 1$,

$$0 < \lambda U_D^p 1(x) \leqslant U_D^p\lambda(x) \leqslant C\big(z(\lambda) + 1\big)$$

whence $\lim_{\lambda\to\infty} z(\lambda) = \infty$. □

Lemma 4.2. *Let $D = \{x \in \mathbb{R}^N \colon |x| < R\}$, where $0 < R \leqslant \infty$, and let u, v be nonnegative radially symmetric functions on D such that u is a supersolution to* (2.3)*, v is a subsolution to* (2.3)*, $v > 0$ on $D \setminus \{0\}$ and*

$$\liminf_{x\to 0} \frac{u(x)}{v(x)} \leqslant 1.$$

Then $u \leqslant v$ everywhere in D.

Proof. Let us fix $0 < \rho < R$ and define

$$a = \frac{u(\rho, 0, \dots, 0)}{v(\rho, 0, \dots, 0)}.$$

To prove Lemma 4.2 we have to show that $a \leqslant 1$. Assuming that $a > 1$ let us consider the function $w = av$ on $B = \{x \in \mathbb{R}^N\colon |x| < \rho\}$. Then

$$\Delta w = a\Delta v \geqslant apv^\gamma \geqslant pw^\gamma \quad \text{in } B,$$

that is, w is a subsolution to (2.3) in B. Since u is a solution to (2.3) in B and $u = w$ on ∂B, we conclude by Lemma 2.1 that $av = w \leqslant u$ on B. Thus

$$a \leqslant \liminf_{x \to 0} \frac{u(x)}{v(x)} \leqslant 1,$$

a contradiction finishing the proof. □

Corollary 4.3. *Let D be as in the previous lemma and consider radially symmetric continuous functions u, v such that $u \geqslant 0$ is a supersolution and $v > 0$ is a subsolution to* (2.3) *in D.*

(a) *If $u(0) \leqslant v(0)$, then $u(x) \leqslant v(x)$ for all $x \in D$.*
(b) *Assume that u and v are even solutions to Eq.* (2.3) *in D. Then u and v coincide everywhere in D provided $u(0) = v(0)$.*

Remark 4.4. Consider a nonnegative radially symmetric continuous function v on the domain $D = \{x \in \mathbb{R}^N\colon |x| < R\}$, where $0 < R \leqslant \infty$. If v is a subsolution to Eq. (2.3) in D, then for all $x, y \in D$,

$$v(x) \leqslant v(y) \quad \text{whenever } |x| \leqslant |y|. \tag{4.1}$$

In fact, a simple application of the classical maximum principle (see, e.g., [9] or [1]) shows that (4.1) holds for every nonnegative radially symmetric subharmonic function on D. Therefore the limit

$$\lim_{|x| \to \infty} v(x)$$

exists in $[0, \infty]$. It is finite if and only if v is bounded.

The following result is already obtained by Lair and Wood [14] for continuous function p. We give here an alternative proof essentially based on the operator U_D^p.

Theorem 4.5. *Assume that the nonnegative locally bounded function p is radially symmetric on $\mathbb{R}^N$, $N \geqslant 3$. Then Eq.* (2.3) *has an entire large solution if and only if*

$$\int_0^\infty r p(r)\, dr = \infty. \tag{4.2}$$

Proof. By Corollary 3.2, Eq. (2.3) has no entire large solution if (4.2) does not hold. So it remains to prove the sufficiency. Suppose that (4.2) holds and let $c > 0$ be a real constant. We claim that Eq. (2.3) has a positive radially symmetric solution u in $\mathbb{R}^N$ satisfying

$$u(0) = c \quad \text{and} \quad \lim_{|x| \to \infty} u(x) = \infty.$$

To prove the claim we define, for every natural $n \geqslant 1$, $D_n := B(0,n)$ to be the ball of center 0 and radius n, and let

$$\lambda_n := \inf\{\lambda \geqslant 0\colon\ U^p_{D_n}\lambda(0) \geqslant c\}.$$

By Lemma 4.1, λ_n is a positive real number satisfying $U^p_{D_n}\lambda_n(0) = c$. On the other hand, since nonnegative constants are supersolutions to (2.3), we conclude by (b2) in Proposition 2.4 that $U^p_{D_n}\lambda_{n+1} \geqslant U^p_{D_{n+1}}\lambda_{n+1}$. Hence

$$U^p_{D_n}\lambda_{n+1}(0) \geqslant U^p_{D_{n+1}}\lambda_{n+1}(0) = c$$

and thereby $\lambda_{n+1} \geqslant \lambda_n$. Let

$$u_n := U^p_{D_n}\lambda_n \quad (n \geqslant 1).$$

Then u_n and u_{n+1} are nonnegative radial solutions to Eq. (2.3) in D_n satisfying $u_n(0) = u_{n+1}(0) = c$. Therefore

$$u_n = u_{n+1} \quad \text{in } D_n$$

by statement (c) in Corollary 4.3. Hence there is a real function u on $\mathbb{R}^N$ such that $u = u_n$ on D_n, and u is a positive radial solution to (2.3) in $\mathbb{R}^N$ satisfying $u(0) = c$. By [10, Theorem 1], u is unbounded. Hence $\lim_{|x|\to\infty} u(x) = \infty$ by the previous remark. □

5. Nonradial perturbation

In this section, we consider the general case, where p is not necessarily radial. We present conditions under which Eq. (2.3) has an entire large solution in $\mathbb{R}^N$ (see Theorem 5.1). Again, let $p \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a nonnegative function in $\mathbb{R}^N$, $N \geqslant 3$. We define

$$p_{\mathrm{osc}}(r) := p^*(r) - p_*(r) \quad (r \geqslant 0), \tag{5.1}$$

where

$$p^*(r) := \sup_{|x|=r} p(x) \quad \text{and} \quad p_*(r) := \inf_{|x|=r} p(x). \tag{5.2}$$

Recently, Lair [12] investigated the case, where p satisfies

$$\int_0^\infty r p_{\mathrm{osc}}(r) \exp\left(\int_0^r t p_*(t)\,dt\right) dr < \infty. \tag{5.3}$$

He proved that, in this case, Eq. (2.3) has an entire large solution in $\mathbb{R}^N$ if and only if

$$\int_0^\infty r p_*(r)\,dr = \infty. \tag{5.4}$$

However, the question of existence or nonexistence of entire large solutions to (2.3), without the additional assumption (5.3), remained open (see also [14]).

We shall prove that, in general, condition (5.4) does not characterize the existence of entire large solutions to the sublinear elliptic equation (2.3). Furthermore, we give a condition weaker than (5.3) under which (5.4) characterizes the existence of such a solution. More precisely, we obtain the following:

Theorem 5.1. *Suppose that* $0 < \gamma < 1$, $N \geqslant 3$, *and*

$$\int_0^\infty r p_{\mathrm{osc}}(r) \left(1 + \int_0^r t p_*(t)\,dt\right)^{\gamma/(1-\gamma)} dr < \infty. \tag{5.5}$$

Then Eq. (2.3) *admits an entire large solution in* $\mathbb{R}^N$ *if and only if* (5.4) *holds.*

Remark 5.2. Lair [12] considered equations of the type $\Delta u = p(x) f(u)$ where p is a nonnegative continuous function on $\mathbb{R}^N$, $N \geqslant 3$, and f is a nondecreasing function on $\mathbb{R}_+$ such that $f(0) = 0$, $f(t) > 0$ for all $t > 0$, and $\Lambda := \sup_{t \geqslant 1} f(t)/t < \infty$. Assuming that

$$\int_0^\infty r p_{\mathrm{osc}}(r) \exp\left(\frac{\Lambda}{N-2} \int_0^r t p_*(t)\,dt\right) dr < \infty,$$

he proved that the existence of large solutions in $\mathbb{R}^N$ is characterized by (5.4). Considering only the case $f(t) = t^\gamma$ where $0 < \gamma < 1$, we show here that this characterization is still true under (5.5). Let us note that condition (5.5) is weaker than condition (5.3). Indeed, if $p_*(r) = c > 0$ for all $r \geqslant 0$, then

$$(5.3) \quad \Leftrightarrow \quad \int_0^\infty \bigl(p^*(r) - c\bigr) r e^{cr^2/2}\,dr < \infty,$$

$$(5.5) \quad \Leftrightarrow \quad \int_0^\infty \bigl(p^*(r) - c\bigr) r \bigl(1 + r^{2\gamma/(1-\gamma)}\bigr)\,dr.$$

Obviously we may choose p such that (5.3) holds but (5.5) does not.

Remarks 5.3.

(1) We define, for every $r \geqslant 0$ and every $x \in \mathbb{R}^N$,

$$V_0(r) := r^2 \int_0^1 \bigl(1 - t^{N-2}\bigr) t p_*(rt)\,dt, \qquad V(x) := \frac{1-\gamma}{N-2} V_0\bigl(|x|\bigr).$$

By Fubini's theorem,

$$\int_0^r s^{1-N}\left(\int_0^s t^{N-1} p_*(t)\,dt\right) ds = \int_0^r t^{N-1} p_*(t)\left(\int_t^r s^{1-N}\,ds\right) dt$$
$$= \frac{1}{N-2} \int_0^r \left(1 - \left(\frac{t}{r}\right)^{N-2}\right) t p_*(t)\,dt = \frac{V_0(r)}{N-2}.$$

Therefore

$$V_0''(r) + \frac{N-1}{r} V_0'(r) = (N-2) p_*(r),$$

which yields that

$$\Delta V = (1-\gamma)p_*\big(|x|\big) \quad \text{in } \mathbb{R}^N.$$

(2) Assume that (5.5) holds. Then $\int_0^\infty r p_{\mathrm{osc}}(r)\,dr < \infty$, and consequently (5.4) holds if and only if

$$\int_{\mathbb{R}^N} \frac{p(x)}{|x|^{N-2}}\,dx = \infty.$$

(3) Since $V_0(r) \leqslant \int_0^r t p_*(t)\,dt$ for every $r \geqslant 0$, condition (5.5) implies that

$$\int_{\mathbb{R}^N} \frac{p_{\mathrm{osc}}(|y|)}{|y|^{N-2}}\big(1 + V^{\gamma/(1-\gamma)}(y)\big) < \infty. \tag{5.6}$$

In order to prove Theorem 5.1, we need the following two lemmas.

Lemma 5.4. *Let $0 < \gamma < 1$ and let u be a nonnegative continuous radially symmetric function satisfying $\Delta u \leqslant p_*(|x|)u^\gamma$ in $\mathbb{R}^N$. Then*

$$u \leqslant 2^{\gamma/(1-\gamma)}\big(u(0) + V^{1/(1-\gamma)}\big) \quad \text{in } \mathbb{R}^N. \tag{5.7}$$

Proof. Let $\varepsilon > 0$, $f := (u(0)+\varepsilon)^{1-\gamma} + V$ and $v := f^{1/(1-\gamma)}$. Then

$$\Delta v = \frac{1}{1-\gamma} f^{\gamma/(1-\gamma)}\Delta f + \frac{\gamma}{(1-\gamma)^2} f^{(2\gamma-1)/(1-\gamma)}|\nabla f|^2 \geqslant p_*\big(|x|\big)v^\gamma.$$

So, since $u(0) \leqslant v(0)$ and $v > 0$ on $\mathbb{R}^N$, $u(x) \leqslant v(x)$ for all $x \in \mathbb{R}^N$ by Corollary 4.3. Thus

$$u \leqslant \big(\big(\varepsilon + u(0)\big)^{1-\gamma} + V\big)^{1/(1-\gamma)} \leqslant 2^{\gamma/(1-\gamma)}\big(\varepsilon + u(0) + V^{1/(1-\gamma)}\big).$$

Letting ε tend to zero we get (5.7). □

Lemma 5.5. *Let Ω be an open subset of $\mathbb{R}^N$ and let u (respectively v) be a subsolution (respectively supersolution) to (2.3) in Ω such that $0 \leqslant u \leqslant v$ in Ω. Then Eq. (2.3) has a solution w in Ω satisfying $u \leqslant w \leqslant v$ in Ω.*

Proof. Choose a regular exhaustion $(D_n)_{n\geqslant 1}$ of Ω and define $w_n := U^p_{D_n}u$ for every $n \geqslant 1$. Then statements (a2) and (b2) in Proposition 2.4 yield that

$$u \leqslant w_n \leqslant w_{n+1} \leqslant v.$$

Therefore, the function w given

$$w(x) = \lim_{n\to\infty} w_n(x)$$

is well defined on Ω, and clearly $u \leqslant w \leqslant v$ in Ω. By Corollary 2.7, w is a solution to Eq. (2.3) in Ω. □

Proof of Theorem 5.1. Obviously (5.5) yields that

$$\int_0^\infty r p_{\mathrm{osc}}(r)\,dr < \infty. \tag{5.8}$$

Assume first that (5.4) does not hold. Then (3.2) holds and hence, by (A_2') in Corollary 3.2, Eq. (2.3) does not admit entire large solutions.

Suppose now that condition (5.4) holds. By Theorem 4.5, there exists a positive continuous radial function v on $\mathbb{R}^N$ such that

$$v(0) = 1 \quad \text{and} \quad \Delta v = p_*\big(|x|\big)v^\gamma.$$

For every $n \geqslant 1$, consider $B_n := B(0,n)$ and $u_n := U_{B_n}^{p^*} v$, i.e., u_n is the unique solution of the boundary value problem

$$\begin{cases} \Delta u_n = p^*(|x|)u_n^\gamma & \text{in } B_n, \\ u_n = v & \text{on } \partial B_n. \end{cases} \tag{5.9}$$

Then, $0 \leqslant u_{n+1} \leqslant u_n \leqslant v$ in B_n in view of (b) in Proposition 2.4. Consequently, the function u defined for every $x \in \mathbb{R}^N$ by $u(x) := \lim_{n\to\infty} u_n(x)$ is a solution to the equation

$$\Delta u = p^*\big(|x|\big)u^\gamma \quad \text{in } \mathbb{R}^N.$$

Moreover, $0 \leqslant u \leqslant v$ in $\mathbb{R}^N$. Hence, by Lemma 5.5, Eq. (2.3) admits an entire solution w such that $u \leqslant w \leqslant v$ in $\mathbb{R}^N$. Thus the proof will be complete if we show that

$$\lim_{|x|\to\infty} u(x) = \infty. \tag{5.10}$$

To that end, we first notice that the uniqueness of the solution to problem (5.9) and the fact that the Laplacian commutes with orthogonal transformations yield that all functions u_n are radial, and thereby the limit function u is also radial on $\mathbb{R}^N$. On the other hand, (5.4) trivially implies that

$$\int_0^\infty r p^*(r)\,dr = \infty.$$

Therefore, (5.10) follows from [10, Theorem 1] provided u is not identically zero in $\mathbb{R}^N$ (see Remark 4.4). By (a1) in Proposition 2.4, we have

$$\begin{aligned} u_n(0) + \int_{B_n} G_{B_n}(0,y)p^*\big(|y|\big)u_n^\gamma(y)\,dy &= H_{B_n}v(0) \\ &= v(0) + \int_{B_n} G_{B_n}(0,y)p_*\big(|y|\big)v^\gamma(y)\,dy. \end{aligned}$$

Together with the inequality $u_n \leqslant v$, this yields that

$$u_n(0) + \int_{B_n} G_{B_n}(0,y)p_{\mathrm{osc}}\big(|y|\big)u_n^\gamma(y)\,dy \geqslant v(0). \tag{5.11}$$

Let $y \in \mathbb{R}^N \setminus \{0\}$. Using (2.1) and Lemma 5.4 we get

$$G_{B_n}(0,y)p_{\mathrm{osc}}\big(|y|\big)u_n^\gamma(y) \leqslant c_N|y|^{2-N}p_{\mathrm{osc}}\big(|y|\big)v^\gamma(y)$$
$$\leqslant C|y|^{2-N}p_{\mathrm{osc}}\big(|y|\big)\big(1+V^{\gamma/(1-\gamma)}(y)\big) := F(y),$$

where C is a positive constant depending only on N and γ. By (5.6), the function F is Lebesgue-integrable on $\mathbb{R}^N$. Therefore, applying the dominated convergence theorem it follows from (5.11) that

$$u(0) + \int_{\mathbb{R}^N} G(0,y)p_{\mathrm{osc}}\big(|y|\big)u^\gamma(y)\,dy \geqslant v(0) > 0.$$

Thus u is not identically zero finishing the proof. $\square$

Remark 5.6. As the proof shows, Theorem 5.1 is still valid provided (5.5) is replaced by (5.6).

In the remainder of this section, we intend to show that hypothesis (5.5) in Theorem 5.1 can not be dropped. We shall first prove that, without (5.5), condition (5.4) is not sufficient for (2.3) to possess an entire large solution in $\mathbb{R}^N$. This is a direct consequence of the following theorem:

Theorem 5.7. *Let ψ be nonnegative continuous function on $\mathbb{R}_+$, (ε_n) be a sequence of positive real numbers, and let (x_n) be a sequence in $\mathbb{R}^N$ such that*

$$0 < |x_n| < |x_{n+1}|, \qquad \lim_{n\to\infty} |x_n| = \infty.$$

Then there exists a continuous function p on $\mathbb{R}^N$ such that:

(i) *For every $x \in \mathbb{R}^N$, $p(x) \geqslant \psi(|x|)$.*
(ii) *For every nonnegative entire solution u to Eq.* (2.3), *$u(x_n) \leqslant \varepsilon_n$ for almost all $n \in \mathbb{N}$.*

Proof. *Step 1.* Let $a \in \mathbb{R}^N$ and let λ, r, ε be positive real numbers. Then there exists a function $q \in \mathcal{C}_c^+(B(a,r))$ such that

$$U^q_{B(a,r)}\lambda(a) \leqslant \varepsilon.$$

Indeed, without loss of generality, we may assume that $a = 0$. Let $B = B(0,r)$ and choose a radial function $\varphi \in \mathcal{C}_c^+(B)$ such that

$$\int_B G_B(0,y)\varphi(y)\,dy \geqslant \frac{\lambda}{\varepsilon^\gamma}. \tag{5.12}$$

Since $u := U^\varphi_B\lambda$ is radially symmetric on B, it follows from Remark 4.4 that $u(y) \geqslant u(0)$ for all $y \in B$. Therefore

$$\lambda = u(0) + \int_B G_B(0,y)\varphi(y)u^\gamma(y)\,dy \geqslant u^\gamma(0)\int_B G_B(0,y)\varphi(y)\,dy,$$

whence $u(0) \leqslant \varepsilon$ by (5.12).

Step 2. We choose $R_n, \rho_n > 0$ $(n \geqslant 1)$ such that

$$|x_n| < R_n < |x_{n+1}| \quad \text{and} \quad \overline{B(x_n,\rho_n)} \subset B(0,R_n) \setminus \overline{B(0,R_{n-1})}$$

(where $R_0 := 0$). We define

$$B_n := B(x_n, \rho_n), \quad V := \bigcup_{i \geqslant 1} B_i, \quad K_n := \{0\} \cup \partial B(0, R_n), \quad D := \mathbb{R}^N \setminus \overline{V}.$$

According to Proposition 2.5, for every $n \geqslant 1$, there exists a constant $C_n > 0$ such that the inequality

$$\sup_{x \in K_n} v(x) \leqslant C_n\bigl(v(0) + 1\bigr) \tag{5.13}$$

holds for every nonnegative solution v to the equation $\Delta v = \psi(|x|)v^\gamma$ in D. Let $\lambda_n := C_n(n+1)$, $n \geqslant 1$. By the first step, we may find a nonnegative continuous function φ_n on $\mathbb{R}^N$ such that $\varphi_n = 0$ outside B_n and $w_n(x_n) \leqslant \varepsilon_n$, where w_n is the unique solution to the boundary value problem

$$\begin{cases} \Delta w_n = \varphi_n(x) w_n^\gamma & \text{in } B_n, \\ w_n = \lambda_n & \text{on } \partial B_n. \end{cases}$$

Now consider the function p defined by

$$p(x) := \psi\bigl(|x|\bigr) + \sum_{i=1}^{\infty} \varphi_i(x) \quad \bigl(x \in \mathbb{R}^N\bigr).$$

Let u be a nonnegative entire solution to (2.3) in $\mathbb{R}^N$ and let $n \geqslant u(0)$. Since $\Delta u = \psi(|x|)u^\gamma$ in D, it follows from (5.13) that $u \leqslant \lambda_n$ in $\partial B(0, R_n)$ whence $u \leqslant \lambda_n$ in $B(0, R_n)$. By (a2) and (c) of Proposition 2.4, $u \leqslant w_n$ in B_n. In particular, $u(x_n) \leqslant w(x_n) \leqslant \varepsilon_n$. $\square$

Moreover, the following example shows that Eq. (2.3) may have a positive large solution in $\mathbb{R}^N$ even if $p_*(r) = 0$ for all $r \geqslant 0$. So, without assuming (5.5), condition (5.4) is not necessary for the existence of an entire large solution to (2.3).

Example 5.8. Let $\varphi(r) := \sin^2(\pi r)$ for every $r \geqslant 0$ and let a be a fixed point in $\mathbb{R}^N$ such that $|a| = 1$. For every $x \in \mathbb{R}^N$, we define q and p by

$$q(x) := \varphi\bigl(|x|\bigr) \quad \text{and} \quad p(x) := q(x+a) = \varphi\bigl(|x+a|\bigr).$$

Then (5.4) fails, but Eq. (2.3) has an entire large solution in $\mathbb{R}^N$.

Indeed, since $\varphi(n) = 0$ for every $n \in \mathbb{N}$, and

$$\bigl\{y \in \mathbb{R}^N \colon |y - a| = r,\ |y| \in \mathbb{N}\bigr\} \neq \emptyset,$$

we conclude that $p_*(r) = 0$ for all $r \geqslant 0$. Hence (5.4) does not hold. To prove that (2.3) possesses an entire large solution in $\mathbb{R}^N$, we observe that

$$\int_0^\infty r\varphi(r)\, dr = \infty,$$

which implies the existence of a positive large solution v to the equation $\Delta v = qv^\gamma$ in $\mathbb{R}^N$. Defining $u(x) := v(x+a)$ for every $x \in \mathbb{R}^N$, we obtain an entire large solution to Eq. (2.3) in $\mathbb{R}^N$.

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